MAT 1322
Lecture 20

29/03/2017

CHAIN RULE CASE II

$z = f(x,y)$ where $x = x(s,t)$
$y = y = y(s,t)$

Observe that z is a function of 2 variables s and t, ∴ 2 partial derivatives

$$\boxed{\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} \end{aligned}}$$

Ex $z = e^x \sin(y)$, $x(s,t) = st^2$ $y(s,t) = s^2t$

$\frac{\partial z}{\partial x} = e^x\sin(y)$ $\frac{\partial z}{\partial y} = e^x\cos(y)$

$\frac{\partial x}{\partial s} = t^2$ $\frac{\partial y}{\partial s} = 2st$

$\frac{\partial x}{\partial t} = 2st$ $\frac{\partial y}{\partial t} = s^2$

$$\frac{\partial z}{\partial s} = e^x\sin(y)(t^2) + e^x\cos(y)(2st)$$

← substitute $x(s,t)$, $y(s,t)$

$$= e^{st^2}\sin(s^2t)(t^2) + e^{st^2}\cos(s^2t)(2st)$$

$$\frac{\partial z}{\partial t} = e^x\sin(y)(2st) + e^x\cos(y)(s^2)$$

$$= e^{st^2}\sin(s^2t)(2st) + e^{st^2}\cos(s^2t)(s^2)$$

IMPLICIT DIFFERENTIATION

CASE 1: $F(x, y(x)) = 0$ ← y is a function in x only!

$$\boxed{y' = \frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}}$$

Ex $x^3 + y^3 = 6xy$ equation in $x, y(x)$

$x^3 + y^3 - 6xy = 0$ so $F(x, y(x)) = x^3 + y^3 - 6xy$

$\frac{\partial F}{\partial x} = 3x^2 - 6y$, $\frac{\partial F}{\partial y} = 3y^2 - 6x$ ∴ $\frac{-3x^2 - 6y}{3y^2 - 6x} = \underline{\frac{-x^2 - 2y}{y^2 - 2x}}$

CASE II: $F(x,y,z)=0$, $z=z(x,y)$

+ z is a function in 2 variables, so 2 partial derivatives

$$\boxed{\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}} \qquad \boxed{\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}}$$

<u>Ex</u> $x^3+y^3+z^3=-6xyz+1$

find F: $x^3+y^3+z^3+6xyz-1=0$

$F(x,y,z)=x^3+y^3+z^3+6xyz-1$

<u>need all 3 partial derivatives:</u>

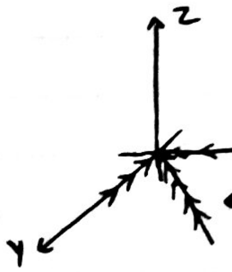
$$\frac{\partial F}{\partial x}=3x^2+6yz \qquad \frac{\partial F}{\partial y}=3y^2+6xz \qquad \frac{\partial F}{\partial z}=3z^2+6xy$$

$$\frac{\partial z}{\partial x}=\frac{-3x^2+6yz}{3z^2+6xy}=\frac{-x^2+2yz}{z^2+2xy} \qquad \frac{\partial z}{\partial y}=-\frac{3y^2+6xz}{3z^2+6xy}=-\frac{y^2+2xz}{z^2+2xy}$$

DIRECTIONAL DERIVATIVES

so far: f_x : der. w.r.t. x; directional derivative along $(1,0)=\vec{u}_x$

f_y : der. w.r.t. y; directional derivative along $(0,1)=\vec{u}_y$



← approach $(0,0)$ along $\vec{u}=(1,1)$, for example

For our computations we had $\vec{u}$ to have length 1

<u>recall</u> length of a vector $\vec{u}=(a,b)$ is given by $L_{\vec{u}}=\sqrt{a^2+b^2}$, so

$\vec{u}_e=\dfrac{(a,b)}{\sqrt{a^2+b^2}}$ is now a unit vector, length 1

<u>recall</u> $f_x(a,b)=\lim\limits_{h\to 0}\dfrac{f(a+h,b)-f(a,b)}{h}$

$f_y(a,b)=\lim\limits_{h\to 0}\dfrac{f(a,b+h)-f(a,b)}{h}$

$f_{\vec{h}}(a,b)=\lim\limits_{h=(h_1,h_2)\to(0,0)}\dfrac{f(a+h_1,b+h_2)-f(a,b)}{\sqrt{h_1^2+h_2^2}}$

} this explains why the formula works

Directional derivative along $\vec{u} = (a,b)$ with $a^2+b^2=1$

$$D_{\vec{u}} f(x,y) = f_x(x,y)\cdot a + f_y(x,y)\cdot b$$

<u>Ex</u> directional derivative of $f(x,y) = x^3 - 3xy + 4y^2$ along line $y=3x$.

First: need a vector

Start w/ $\vec{u} = (1,3)$ and make into length 1

$$\vec{u}_e = \frac{(1,3)}{\sqrt{1^2+3^2}} = \frac{1}{\sqrt{10}}(1,3) = \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$$

Now: use formula

$$D_{\vec{u}_e} f(x,y) = f_x(x,y)\left(\frac{1}{\sqrt{10}}\right) + f_y(x,y)\cdot\left(\frac{3}{\sqrt{10}}\right)$$

$f_x = 3x^2 - 3y$

$f_y = -3x + 8y$

$$D_{\vec{u}_e} f(x,y) = (3x^2-3y)\left(\frac{1}{\sqrt{10}}\right) + (-3x+8y)\left(\frac{3}{\sqrt{10}}\right)$$

$$= \frac{1}{\sqrt{10}}(3x^2 - 3y - 9x + 24y)$$

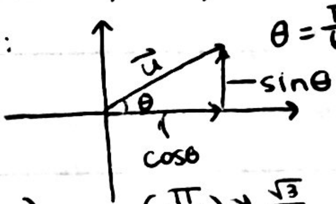
$$= \frac{1}{\sqrt{10}}(3x^2 - 9x + 21y)$$

<u>Ex 2</u>

Approach $f(x,y) = x^3 - 3xy + 4y^2$ from an angle $\theta = \frac{\pi}{6}$

first find the vector:

$\theta = \frac{\pi}{6}$



$\vec{u} = (\cos(\frac{\pi}{6}), \sin(\frac{\pi}{6}))$

* length is already 1

$(\sin^2(x) + \cos^2(x) = 1)$

$f_x = 3x^2 - 3y$

$f_y = -3x + 8y$

$$D_{\vec{u}} f(x,y) = (3x^2-3y)\cos\left(\tfrac{\pi}{6}\right) \to \tfrac{\sqrt{3}}{2}$$
$$+ (-3x+8y)\sin\left(\tfrac{\pi}{6}\right) \to \tfrac{1}{2}$$

$$= \frac{1}{2}(\sqrt{3}\cdot 3x^2 - \sqrt{3}\cdot 3y - 3x + 8y)$$

$$= \frac{1}{2}(\sqrt{3}\cdot 3x^2 - 3x + y(8 - \sqrt{3}\cdot 3))$$

If we want slope at point $(2,4)$:

$$D_{\vec{u}} f(2,4) = \frac{1}{2}(\sqrt{3}\cdot 3(2)^2 - 3(2) + 4(8 - \sqrt{3}\cdot 3))$$

$$= \frac{1}{2}(\cancel{\sqrt{3}\cdot 12} - 6 + 32 - \cancel{12\sqrt{3}})$$

$$= \frac{26}{2}\ \boxed{= 13}$$

GRADIENT VECTOR

- $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

$$= \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

$\hat{\imath}$ $\hat{\jmath}$

The gradient vector is the vector with components the partial derivatives.

Theorem

$D_{\vec{u}} f(x,y)$ is maximized if $\vec{u} = \nabla f(x,y)$

↳ maximizes slope, not function!

Ex max rate of change at $(2,0)$ of $f(x,y) = xe^y$

$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle$

at $(2,0)$: $\nabla f(2,0) = \langle e^0, 2e^0 \rangle = \langle 1, 2 \rangle$

→ length 1: $\vec{u}_e = \frac{1}{\sqrt{1^2+2^2}} \langle 1,2 \rangle = \frac{1}{\sqrt{5}} \langle 1,2 \rangle$

Now we need directional derivative along $\vec{u}$:

$$D_{\vec{u}} f(x,y) = f_x \frac{1}{\sqrt{5}} + f_y \frac{2}{\sqrt{5}} = \frac{e^y}{\sqrt{5}} + \frac{2xe^y}{\sqrt{5}}$$

now fill in $(2,0)$: $D_{\vec{u}} f(2,0) = \frac{1}{\sqrt{5}} + \frac{2 \cdot 2 \cdot e^0}{\sqrt{5}} = \frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}}$

$$= \frac{5}{\sqrt{5}} \boxed{= \sqrt{5}}$$